GROTHENDIECK-TEICHMÜLLER THEORY AS A SPECIES OF TEICHMÜLLER THEORY [JOINT WORK IN PROGRESS WITH TSUJIMURA] (NGR2025 VERSION)

Shinichi Mochizuki (RIMS, Kyoto University)

October 2025

https://www.kurims.kyoto-u.ac.jp/~motizuki/GT%20as%20Tch%20(NGR2025%20version).pdf

§1. **Ring-theoretic interpretation of complex Teichmüller theory**

- Review of classical complex Teichmüller theory:
  (cf. [EssLgc], Example 3.3.1)

  Recall the most *fundamental deformation of complex structure* in classical complex Teichmüller theory: for $\lambda \in \mathbb{R}_{>1}$,

$$\Lambda : \mathbb{C} \quad \to \quad \mathbb{C}$$

$$\mathbb{C} \quad \ni \; z = x + iy \quad \mapsto \quad \zeta \; = \; \xi + i\eta \; \stackrel{\mathrm{def}}{=} \; \lambda \cdot x + iy \; \in \quad \mathbb{C}$$

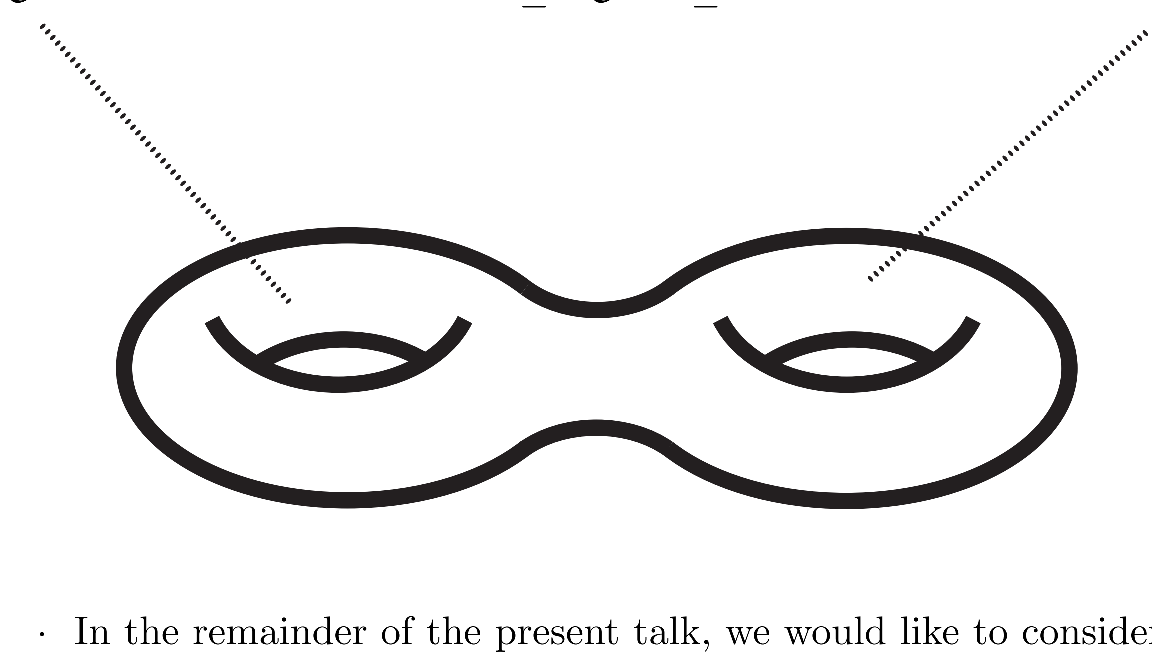
— where $x, y \in \mathbb{R}$.

- Classical complex Teichmüller theory on Riemann surfaces:
  More generally, on a *single (oriented) topological surface* S, we can start with one *holomorphic structure* $\mathfrak{H}_1$ on S and a *square differential* relative to $\mathfrak{H}_1$, then form the *Teichmüller dilation*, or *Teichmüller map*, obtained by deforming (i.e., in the fashion described above) the canonical holomorphic coordinate obtained by integrating the square root of the square differential (along paths) so as to obtain a *new holomorphic structure* $\mathfrak{H}_2$.

  Next, for $i = 1, 2$, write
  $\mathcal{O}^{\mathfrak{H}_i}$ for the sheaf of *holomorphic functions* on S, rel. to $\mathfrak{H}_i$;
  $\mathcal{O}^{\mathfrak{R}}$ for the sheaf of (complex valued) *real analytic fns.* on S.

  Here, we note that for connected open subsets $U \subseteq S$, $\mathcal{O}^{\mathfrak{R}}(U)$ is a *domain*, i.e., unlike the case with continuous or $\mathcal{C}^\infty$-functions. That is to say, $\mathcal{O}^{\mathfrak{R}}$ is in some sense *close* to being like $\mathcal{O}^{\mathfrak{H}_i}$, for $i = 1, 2$, but still *suff'ly large* as to allow one to obtain *embeddings* in the *common container* $\mathcal{O}^{\mathfrak{R}}$:

$$\mathcal{O}^{\mathfrak{H}_1} \qquad \subseteq \quad \mathcal{O}^{\mathfrak{R}} \quad \supseteq \qquad \mathcal{O}^{\mathfrak{H}_2}$$



- In the remainder of the present talk, we would like to consider various *arithmetic analogues* of the function theory discussed above in the complex case.

§2. The case of inter-universal Teichmüller theory (IUT)

· A more detailed exposition of IUT may be found in

· the *survey texts* [Alien], [EssLgc] (cf. also [IUTchI-IV], [IUAni1], [IUAni2]), well as in
· the *videos/slides* available at the following URLs:
(cf. also my series of *DWANGO LECTURES* on IUT
— URLs available at request!):

```
https://www.kurims.kyoto-u.ac.jp/~motizuki/ExpHoriz
      IUT21/WS3/ExpHorizIUT21-InvitationIUT-notes.html
```

```
https://www.kurims.kyoto-u.ac.jp/~motizuki/ExpHoriz
      IUT21/WS4/ExpHorizIUT21-IUTSummit-notes.html
```

· Let R be an *integral domain* (e.g., $\mathbb{Z} \subseteq \mathbb{Q}$) equipped with the action of a *group* G, $(\mathbb{Z} \ni)\ N \geq 2$. For simplicity, assume that $N = 1 + \cdots + 1 \neq 0 \in R$; R has *no nontrivial N-th roots of unity*. Write $R^{\triangleright} \subseteq R$ for the *multiplicative monoid* $R \setminus \{0\}$. Then let us observe that the *N-th power map* on $R^{\triangleright}$ determines an *isomorphism of multiplicative monoids* equipped with actions by G

$$G \curvearrowright \ R^{\triangleright} \quad \stackrel{\sim}{\to} \quad (R^{\triangleright})^N \ (\subseteq R^{\triangleright}) \ \curvearrowleft \ G$$

that does *not arise* from a *ring homomorphism*, i.e., it is clearly *not compatible* with *addition* (cf. our assumption on N!).

· Let ${}^{\dagger}R$, ${}^{\ddagger}R$ be *two distinct copies* of the integral domain R, equipped with respective actions by *two distinct copies* ${}^{\dagger}G$, ${}^{\ddagger}G$ of the group G. We shall use similar notation for objects with labels "†", "‡" to the previously introduced notation. Then one may use the *isomorphism of multiplicative monoids* arising from the *N-th power map* discussed above to *glue* together

$${}^{\dagger}G \ \curvearrowright \ {}^{\dagger}R \supseteq ({}^{\dagger}R^{\triangleright})^N \quad \stackrel{\sim}{\leftarrow} \quad {}^{\ddagger}R^{\triangleright} \subseteq {}^{\ddagger}R \ \curvearrowleft \ {}^{\ddagger}G$$

. . . where the notion of a *gluing* may be understood
· as a *quotient set* via identifications, or (preferably)
· as an *abstract diagram* (cf. graphs of groups/anabelioids!)

the *ring* ${}^{\dagger}R$ to the *ring* ${}^{\ddagger}R$ along the *multiplicative monoid* $({}^{\dagger}R^{\triangleright})^N \stackrel{\sim}{\leftarrow} {}^{\ddagger}R^{\triangleright}$. This gluing is *compatible* with the respective actions of ${}^{\dagger}G$, ${}^{\ddagger}G$ relative to the isomorphism ${}^{\dagger}G \stackrel{\sim}{\to} {}^{\ddagger}G$ given by forgetting the labels "†", "‡", but, since the N-th power map is *not compatible* with *addition* (!), this isomorphism ${}^{\dagger}G \stackrel{\sim}{\to} {}^{\ddagger}G$ may be regarded either as an isomorphism of ("*coric*", i.e., *invariant* with respect to the N-th power map) *abstract groups* (cf. the notion of "*inter-universality*", as discussed in [EssLgc], §3.2, §3.8!) or as an isomorphism of groups equipped with actions on certain *multiplicative monoids*, but *not* as an isomorphism of ("*Galois*" — cf. the *inner automorphism indeterminacies* of SGA1!) groups equipped with actions on *rings/fields*.

· The problem of *describing (certain portions of the) ring structure* of $^{\dagger}R$ in terms of the *ring structure* of $^{\ddagger}R$ — in a fashion that is *compatible* with the *gluing* and via a *single* algorithm that may be applied to the *common* (cf. *logical AND $\wedge$!*) *glued data* to reconstruct *simultaneously* (certain portions of) the ring structures of *both* $^{\dagger}R$ and $^{\ddagger}R$, up to suitable relatively mild *indeterminacies* (cf. the theory of *crystals*!) — seems (at first glance/in general) to be *hopelessly intractable* (cf. the case of $\mathbb{Z}$)!

  . . . where we note that here, considering *portions* is important because we want to *decompose* the above diagram up into *pieces* so that we can consider *symmetry* properties involving these pieces!

  One well-known elementary example: when $N = p$, working *modulo p* (cf. *indeterminacies*, analogy with *crystals*!), where there is a *common ring structure* that is *compatible* with the *p-th power map*!

  Another important example: Faltings' proof of *invariance* of *height* of elliptic curves under *isogeny*, under the assumption of the existence of a *global multiplicative subspace* (cf. [ClsIUT], §1; [EssLgc], Example 3.2.1)!

  . . . This is precisely what is *achieved in IUT* by means of the *multiradial representation for the Θ-pilot* via

  · *anabelian geometry* (cf. the *abstract groups* $^{\dagger}G$, $^{\ddagger}G$!);
  · *the p-adic/complex logarithm, Galois eval. of theta functions*;
  · *Kummer theory*, to relate *Frob.-/étale-like* versions of objects.

· *Main point*:

  The *multiplicative monoid* and *abstract group* structures (but *not* the ring structures!) are *common* (cf. "*logical AND $\wedge$!*") to †, ‡ and can be used as the *input data* for an algorithm to construct the *multirad. rep. for the Θ-pilot*, i.e., a *common container* for the distinct *ring strs.* (i.e., "*arith. hol. strs.*") †, ‡

$$^{\dagger}R \quad \subseteq \quad \Big(\text{multirad. rep. for the } \Theta\text{-pilot}\Big) \quad \supseteq \quad ^{\ddagger}R$$

· When $R = \mathbb{Z}$ (or, in fact, more generally, the *ring of integers* "$\mathcal{O}_F$" of a number field F — cf. the multiplicative *norm map* $\mathrm{N}_{F/\mathbb{Q}} : F \to \mathbb{Q}$), one may consider the "*height/log-volume*"

$$\log(|x|) \in \mathbb{R}$$

  for $0 \neq x \in \mathbb{Z}$. Then the *N-th power map* of (i), (ii) corresponds, after passing to *heights*, to *multiplying real numbers by N*; the *multiradial algorithm* corresponds to saying that the height is *unaffected (up to a mild error term!)* by *multiplication by N*, hence that the *height is bounded*!

· In the case of IUT, the *multirad. rep. for the Θ-pilot* is obtained by means of a sort of "*analytic continuation*" along a certain "*infinite H*" of the *log-theta-lattice* (cf. the discussion surrounding [EssLgc], §3.3, (InfH))

... where

· the *Θ-link* between distinct ring strs. "•" corresponds to the N-th power map discussed in the present §2, while

· the *$\mathfrak{log}$-link* locally at nonarchimedean valuations looks like the p-adic logarithm between distinct ring strs. "•";

· the *descent operations* revolve around the establishment of certain *coricity/symmetry* properties.

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \vdots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots & & \bullet & & \bullet \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots & \supseteq & \bullet & \xrightarrow{\Theta} & \bullet \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots & & \bullet & & \bullet \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
 & & \vdots & & \vdots & & & & \vdots & & \vdots
\end{array}$$

$$(0,0) \overset{(\mathrm{Stp1})}{\rightsquigarrow} (0,\circ) \overset{(\mathrm{Stp2})}{\rightsquigarrow} (0,\circ)^{\vdash} \overset{(\mathrm{Stp3})}{\rightsquigarrow} (0,0)^{\vdash} \overset{(\mathrm{Stp3})}{\Longleftrightarrow} (1,0)^{\vdash}$$

— which involves a gradual introduction via "*descent*" operations of "*fuzzifications*", corresponding to *indeterminacies* (cf. the discussion of [EssLgc], §3.10).

· At a more technical level, the *multirad. rep. for the Θ-pilot* is obtained by constructing *invariants* with respects to the *$\mathfrak{log}$-link*, which has the effect of *juggling addition and multiplication* — i.e., juggling the *dilated* and *non-dilated* portions of the *ring strs.* — and, as a result, effects a sort of "*miraculous rotation*" (the discussion of [EssLgc], §3.11)

$$\begin{array}{ccc}
 & & \bullet \\
 & \text{“}\curvearrowright\text{”} & \uparrow \mathfrak{log} \\
\bullet & \xrightarrow{\Theta} & \bullet
\end{array}$$

of the

· "*mysterious log-volume-dilating Θ-link gap*" (between the domain/codomain of the Θ-link) onto the

· "*harmless log-volume-preserving $\mathfrak{log}$-link gap*" (between the domain/codomain of the $\mathfrak{log}$-link) !

§3. GT via genus zero quotients
(cf. [CbGT]; [CbGal]; [ArGT], §2, §3)

- The following result in *combinatorial anabelian geometry* on the *faithfulness* of the natural outer action of GT on the *genus zero quotient* of the geometric fundamental group of the tripod is the *key technical result* that underlies our new results on GT:

Let
- K be a field of *characteristic* 0;
- $\overline{K}$ an *algebraic closure* of K;
- X a *hyperbolic curve* of genus 0 with r cusps over K.

Write Π_X for the *étale fundamental group* of $X_{\overline{K}} \stackrel{\text{def}}{=} X \times_K \overline{K}$;

$$(\Pi_X \twoheadrightarrow) \quad \Pi_X^0 \quad \stackrel{\text{def}}{=} \quad \Pi_X / \left(\bigcap_H H \right)$$

— where $H \subseteq \Pi_X$ ranges over the open subgroups of Π_X corresponding to *genus* 0 finite étale coverings of $X_{\overline{K}}$, and we recall that $\mathrm{Ker}(\Pi_X \twoheadrightarrow \Pi_X^0) \neq \{1\}$ (although, by a classical result of Ihara-Anderson, "$= \{1\}$" holds in the *pro-l case*) — for the *genus zero quotient* of Π_X;

$$\mathrm{Out}^{\mathrm{C}}(\Pi_X)$$

for the group of outer automorphisms of Π_X that map *cuspidal inertia subgroups* to cuspidal inertia subgroups;

$$\mathrm{Out}^{\mathrm{C}}(\Pi_X) \quad \to \quad \mathrm{Out}(\Pi_X^0)$$

for the natural homomorphism (since elements of $\mathrm{Out}^{\mathrm{C}}(\Pi_X)$ stabilize the class of open subgroups "H" considered above).

Now suppose that the *type* $(0, r)$ of X is equal to $(0, 3)$, i.e., that X is a *tripod*, and that $K = \mathbb{Q}$. Thus, as is well-known ("Belyi injectivity"; the definition of GT), we have natural *injections*

$$G_{\mathbb{Q}} \quad \hookrightarrow \quad \mathrm{GT} \quad \hookrightarrow \quad \mathrm{Out}^{\mathrm{C}}(\Pi_X)$$

such that the composite homomorphism

$$G_{\mathbb{Q}} \quad \to \quad \mathrm{Out}(\Pi_X^0)$$

is *injective* (cf. [Wtbe]). Then the *key technical result* referred to above generalizes this injectivity to the case of GT:

Theorem A [Expected] (Faithfulness of natural outer action of GT on the genus zero quotient). *The natural composite homomorphism*

$$\mathrm{GT} \quad \to \quad \mathrm{Out}(\Pi_X^0)$$

is injective (cf. [ArGT], §3).

The proof of Theorem A yields an *independent proof* of the main result of [Wtbe] referred to above and involves applying highly technical results from *combinatorial anabelian geometry* (cf. [CbGT], [CbGal]) concerning the *sequence of configuration space groups* (i.e., étale fundamental groups of the base-change to $\overline{K}$ of various *configuration spaces* X_n associated to X)

$$\ldots \twoheadrightarrow \Pi_{X_{n+1}} \twoheadrightarrow \Pi_{X_n} \twoheadrightarrow \Pi_{X_{n-1}} \twoheadrightarrow \ldots$$

and the associated *sequence of fiber subgroup-preserving outer automorphism groups*

$$\ldots \overset{\sim}{\to} \mathrm{Out}^{\mathrm{F}}(\Pi_{X_{n+1}}) \overset{\sim}{\to} \mathrm{Out}^{\mathrm{F}}(\Pi_{X_n}) \overset{\sim}{\to} \mathrm{Out}^{\mathrm{F}}(\Pi_{X_{n-1}}) \overset{\sim}{\to} \ldots$$

for *arbitrarily large positive integers* $n \geq 3$ (cf. [CbGT], Corollary C). Here, we note that the passage $X_n \rightsquigarrow X_{n+1}$ corresponds, at "*toral*" (i.e., "*multiplicative*"!) nodes, to a passage to *tripods* (i.e., which involve "*additive*" symmetries $t \mapsto 1-t$), hence may be thought of as a sort of *combinatorial* analogue of the *p-adic logarithm* — cf. the role played by

> the *vertical columns of* $\mathfrak{log}$*-links/absolute p-adic anabelian geometry* in the construction of the *multiradial rep./"miraculous rotation"*
>
> $$\Theta\text{-link } \text{“} \curvearrowright \text{”} \ \mathfrak{log}\text{-link}$$
>
> of IUT in §2, which is obtained by forming *invariants* with respect to the $\mathfrak{log}$*-link*, which constitutes a rotation
>
> $$\textit{addition } \text{“}\boxplus\text{”} \qquad \text{“} \curvearrowright \text{”} \qquad \textit{multiplication } \text{“}\boxtimes\text{”}.$$

The other main technical tool used in the proof of Theorem A is the following *elementary variant* of (the argument in the first half of the proof of) *Belyi's Theorem*:

> Write $L \overset{\text{def}}{=} \mathbb{Q}(t)$, where t is an *indeterminate*. Let $\overline{L}$ be an *algebraic closure* of L, $f \in \overline{L}$. Thus, f may be thought of, via the standard coordinate on the projective line, as a point $\in \mathbb{P}^1_L(\overline{L})$. Then there exists a *"Belyi map"*
>
> $$\mathbb{P}^1_L \to \mathbb{P}^1_L$$
>
> — i.e., a dominant morphism over L that maps the point corresponding to f to an *L-rational point* of $\mathbb{P}^1_L$ and, moreover, is *unramified* outside a finite set of *L-rational points* of $\mathbb{P}^1_L$.

§4. **Decomposition groups and function spaces**
(cf. [CbGT]; [CbGal]; [RNSPM]; [ArGT], §4, §5)

- We maintain the notation of Theorem A of §3 and consider *distinct conjugates* of $G_{\mathbb{Q}}$ inside GT ($\ni \sigma, \tau$)

$$\begin{array}{ccccc} G_{\mathbb{Q}}^{\sigma} & \subseteq & \mathrm{GT} & \supseteq & G_{\mathbb{Q}}^{\tau} \\ (\rightsquigarrow \ \overline{\mathbb{Q}}^{\sigma}) & & & & (\rightsquigarrow \ \overline{\mathbb{Q}}^{\tau}) \end{array}$$

  ... that is to say, *distinct ring theories* (where "$\rightsquigarrow$" denotes *canonical anabelian reconstruction*, as in [CbGal]) that are related by a *mysterious non-ring-theoretic* — i.e., *purely combinatorial/group-theoretic* — *link*;
  ... the (possible) *non-algebraicity* of $\sigma \cdot \tau^{-1}$ may be understood as a sort of "*profinite dilation*" that plays the role of the *complex dilations* of §1, or, alternatively, of the *N-th power map/Θ-link* of §2.

- Just as (noncuspidal) points $\in \mathbb{P}^1_{\mathbb{Q}}(\overline{\mathbb{Q}})$ determine (up to Π_X-conjugacy) — i.e., by considering stabilizers of liftings of these points to points of universal profinite étale coverings of X — *($G_{\mathbb{Q}}$-)decomposition groups*

$$D_{\mathbb{Q}} \quad \subseteq \quad \Pi_X \overset{\text{out}}{\rtimes} G_{\mathbb{Q}} \quad \overset{\text{def}}{=} \quad \mathrm{Aut}(\Pi_X) \times_{\mathrm{Out}(\Pi_X)} G_{\mathbb{Q}}$$

  that map *isomorphically* to open subgroups of $G_{\mathbb{Q}}$, we would like to consider "GT-*decomposition groups*" (or, more generally, "*G-decomposition groups*" for some closed subgroup $G \subseteq$ GT, i.e., by restricting from GT to G)

$$(D_G \subseteq) \ \ D_{\mathrm{GT}} \quad \subseteq \quad \Pi_X \overset{\text{out}}{\rtimes} \mathrm{GT} \quad \overset{\text{def}}{=} \quad \mathrm{Aut}(\Pi_X) \times_{\mathrm{Out}(\Pi_X)} \mathrm{GT}$$

  that map *isomorphically* to open subgroups of GT, by applying the technique of *combinatorial arithmetic Belyi cuspidalizations* (which may be constructed in a *purely combin./group-theoretic* fashion — cf. [CbGal], §3):

$$\begin{array}{ccc} U & \overset{\text{open imm.}}{\hookrightarrow} & X \\ \big\downarrow{\scriptstyle \text{Belyi map}} & & \\ X & & \end{array}$$

  ... which leads us to the following ...

· *Fundamental Injectivity/Conjugate Synchronization Problem*: Given an *arithmetic field* $F \subseteq \overline{\mathbb{Q}}$ (such as $\mathbb{Q}$ or $\overline{\mathbb{Q}} \cap \mathbb{Q}_p$) such that $G_F \subseteq G$ ($\subseteq GT$), is the *restriction map*

$$D_G \quad \mapsto \quad D_G|_{G_F}$$

injective?

... $\overset{\text{(essentially)}}{\iff}$ (since D_G is det'd by $\{D_G|_{G_F^\sigma}\}_{\sigma \in G}$) are the

$$D_G|_{G_F^\sigma}$$

synchronized, as $\sigma \in G$ varies?

· *Construction*: If this *FICSP-property* ($\Longleftarrow$ *COF-property* of [CbGal], §3) holds for G, then for any *genus 0* fin. étale covering

$$Y_{\overline{\mathbb{Q}}} \to X_{\overline{\mathbb{Q}}},$$

one can use G-decomposition groups to construct

· the *set of points* (i.e., $Y_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}) \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \cup \{\infty\}$) and
· the *field structure* (via the *$\mathfrak{S}_3$ symmetries*, i.e., $t \mapsto t^{-1}$, $t \mapsto 1-t$, etc.) on this set of points (i.e., on $\overline{\mathbb{Q}}$),

hence also a natural *function space*

$$\mathrm{Fn}(Y_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}), \overline{\mathbb{Q}}),$$

together with a *subring of algebraic functions* (by considering *standard coord. fns.* — cf. the "*degree one*" *theta fn.* in IUT!) in this function space and a natural *$\Pi_X^0 \overset{\text{out}}{\rtimes} G$-action* (i.e., if we allow the genus 0 finite étale covering $Y_{\overline{\mathbb{Q}}} \to X_{\overline{\mathbb{Q}}}$ to *vary*).

... That is to say, we obtain that the natural outer action of G on Π_X^0 has *algebraic monodromy*, i.e., that the natural homomorphism $G \to \mathrm{Out}(\Pi_X^0)$ *factors* through $G_{\mathbb{Q}}$, which, by Theorem A of §3, implies that ($G_F \subseteq$) $G \subseteq G_{\mathbb{Q}}$.
... In particular, by applying this argument to *closed subgroups* $G \subseteq \mathrm{GT}$ that contain $G_{\mathbb{Q}}$, we obtain the following result:

Theorem B [Expected] (Concise characterization of $G_{\mathbb{Q}}$ in GT). *Let $G \subseteq \mathrm{GT}$ be a closed subgroup that contains $G_{\mathbb{Q}}$ and satisfies the above FICSP-property ($\Longleftarrow$ COF-property of [CbGal], §3). Then (cf. [ArGT], §4)*

$$G \quad = \quad G_{\mathbb{Q}}.$$

That is to say, we obtain a *much simpler/more concise characterization* of $G_{\mathbb{Q}}$ in GT (i.e., by comparison to the characterizations obtained in [CbGal]) via a *single, relatively simple condition*, namely, the *FICSP-property*. (Of course, the ideal result would be "*zero conditions*", i.e., $G_{\mathbb{Q}} = \mathrm{GT}$, but this has not yet been achieved!)

· Next, fix a <u>*prime number*</u> p. For $\Box \in \{\dagger, \ddagger\}$, let

  · $K^{\Box}$ be a *finite extension* of $\mathbb{Q}_p$;
  · $\overline{K}^{\Box}$ an *algebraic closure* of $K^{\Box}$;
  · $X^{\Box}$ a *hyperbolic curve* of type $(0, r^{\Box})$ over $K^{\Box}$.

For $n \geq 1$, write $X^{\Box}_n$ for the *n-th configur. space* assoc'd to $X^{\Box}$;

$$X^{\Box}_{\overline{K}^{\Box}} \stackrel{\mathrm{def}}{=} X^{\Box} \times_{K^{\Box}} \overline{K}^{\Box}, \qquad (X^{\Box}_n)_{\overline{K}^{\Box}} \stackrel{\mathrm{def}}{=} X^{\Box}_n \times_{K^{\Box}} \overline{K}^{\Box};$$

$\Pi_{X^{\Box}}$, $\Pi_{X^{\Box}_n}$ for the respective *étale fundamental groups* of $X^{\Box}_{\overline{K}^{\Box}}$, $(X^{\Box}_n)_{\overline{K}^{\Box}}$. Fix an *isomorphism of profinite groups*

$$\alpha : \Pi_{X^{\dagger}} \xrightarrow{\sim} \Pi_{X^{\ddagger}}.$$

Then we shall say that α is <u>*cuspidalizable*</u> if, for some integer $n \geq 2$, there exists an *isomorphism of profinite gps.* $\Pi_{X^{\dagger}_n} \xrightarrow{\sim} \Pi_{X^{\ddagger}_n}$ that *lifts* α, rel. to the morphisms $\Pi_{X^{\Box}_n} \twoheadrightarrow \Pi_{X^{\Box}}$ (for $\Box \in \{\dagger, \ddagger\}$) induced by the natural projections to the first factor. We shall say that α is <u>*graphic*</u> if it induces a bijection between the resp. collections of *decomposition groups* of irred. comps./nodes of the special fibers of stable models of corresponding (relative to α) finite étale coverings of $\Pi_{X^{\dagger}}$, $\Pi_{X^{\ddagger}}$. If $r^{\dagger} = 3$, then we shall write

$$\mathrm{GT}_p \quad \subseteq \quad \mathrm{GT} \quad (\subseteq \quad \mathrm{Out}(\Pi_{X^{\dagger}}))$$

for the subgroup of <u>*graphic*</u> outer automorphisms, i.e., in essence (cf. [RNSPM], Theorem G), <u>*Yves André's p-adic version*</u> of GT. Then it follows from the theory of <u>*resolution of nonsingularities (RNS)*</u> (cf. [RNSPM], which builds on earlier results due to Akio Tamagawa and Emmanuel Lepage) that

GT_p satisfies the <u>*COF-*</u> ($\Rightarrow$ <u>*FICSP-*</u>) <u>*property*</u>.

In particular, it follows from the theory of [ArGT], §4, §5, that the following results — the first of which (and, to a lesser extent, the second, as well) settles an <u>*important outstanding question*</u> of <u>*André*</u> (i.e., the p-adic analogue of "$G_{\mathbb{Q}} = \mathrm{GT}$") that dates back to around the year 2000 — hold:

Corollary C [Expected] (Algebraicity of GT_p monodromy). *It holds (cf. [ArGT], §4) that* $G_{\mathbb{Q}_p} = \mathrm{GT}_p$.

Corollary D [Expected] (Tempered absolute anabelian result). *There is a <u>natural bijective</u> correspondence (cf. [ArGT], §5) between the set of <u>graphic cuspidalizable</u> outer automorphisms of profinite groups*

$$\Pi_{X^{\dagger}} \xrightarrow{\sim} \Pi_{X^{\ddagger}}$$

(i.e., where one does <u>not</u> consider compatibility with any outer Galois actions!) and the set of isomorphisms of $\mathbb{Q}_p$*-schemes*

$$X^{\dagger}_{\overline{K}^{\dagger}} \xrightarrow{\sim} X^{\ddagger}_{\overline{K}^{\ddagger}}.$$

· The theory discussed above may be summarized as follows:

<u>$\mathbb{C}$Tch</u>	<u>IUT</u>	<u>GT</u>
distinct holomorphic structures $\mathcal{O}^{\mathfrak{H}_i}$, for $i = 1, 2$, on **same** underlying topological surface	**distinct ring/arithmetic holomorphic structures** on opposite sides of the **Θ-link**	**distinct ring structures** corresponding to distinct **conjugates** of $G_{\mathbb{Q}}$ inside GT
embedding of $\mathcal{O}^{\mathfrak{H}_i}$'s into **common container/ domain** $\mathcal{O}^{\mathfrak{R}}$ via **Teichmüller maps**	**multiradial rep.**, up to *mild indets.*, yields **common container** for distinct ring/arith. hol. strs. via **Galois evaluation**, **miraculous rotation** Θ-link " $\curvearrowright$ " $\mathfrak{log}$-link involving $\mathfrak{log}$-*invars.* via a rotation $\boxtimes$ " $\curvearrowright$ " $\boxplus$, **absolute *p*-adic anab. geo.** (cf., especially, (Ind3)), (deg. 1) **étale theta function**	**embedding** of distinct ring strs. into Fn$(-,-)$ via analysis of GT-**dec. groups**, together with **embedding** GT $\hookrightarrow$ Out(Π^0_X) via a **combinatorial rotation** $\boxtimes$ " $\curvearrowright$ " $\boxplus$ (i.e., inf. tower of **config. spaces**), **combinatorial anab. geo.**, **coord. fns.**

References

[IUAni1] E. Farcot, I. Fesenko, S. Mochizuki, *The Multiradial Representation of Inter-universal Teichmüller Theory*, animation available at the following URL:

`https://www.kurims.kyoto-u.ac.jp/~motizuki/IUT-animation-Thm-A-black.wmv`

[IUAni2] E. Farcot, I. Fesenko, S. Mochizuki, *Computation of the log-volume of the q-pilot via the multiradial representation*, animation available at the following URL:

`https://www.kurims.kyoto-u.ac.jp/~motizuki/2020-01%20Computation%20of%20q-pilot%20(animation).mp4`

[IUTchI] S. Mochizuki, Inter-universal Teichmüller Theory I: Construction of Hodge Theaters, *Publ. Res. Inst. Math. Sci.* **57** (2021), pp. 3-207.

[IUTchII] S. Mochizuki, Inter-universal Teichmüller Theory II: Hodge-Arakelov-theoretic Evaluation, *Publ. Res. Inst. Math. Sci.* **57** (2021), pp. 209-401.

[IUTchIII] S. Mochizuki, Inter-universal Teichmüller Theory III: Canonical Splittings of the Log-theta-lattice, *Publ. Res. Inst. Math. Sci.* **57** (2021), pp. 403-626.

[IUTchIV] S. Mochizuki, Inter-universal Teichmüller Theory IV: Log-volume Computations and Set-theoretic Foundations, *Publ. Res. Inst. Math. Sci.* **57** (2021), pp. 627-723.

[Alien] S. Mochizuki, The Mathematics of Mutually Alien Copies: from Gaussian Integrals to Inter-universal Teichmüller Theory, *Inter-universal Teichmuller Theory Summit 2016*, *RIMS Kōkyūroku Bessatsu* **B84**, Res. Inst. Math. Sci. (RIMS), Kyoto (2021), pp. 23-192; available at the following URL:

`https://www.kurims.kyoto-u.ac.jp/~motizuki/Alien%20Copies,%20Gaussians,%20and%20Inter-universal%20Teichmuller%20Theory.pdf`

[EssLgc] S. Mochizuki, *On the essential logical structure of inter-universal Teichmüller theory in terms of logical AND "$\wedge$"/logical OR "$\vee$" relations: Report on the occasion of the publication of the four main papers on inter-universal Teichmüller theory*, preprint available at the following URL:

`https://www.kurims.kyoto-u.ac.jp/~motizuki/Essential%20Logical%20Structure%20of%20Inter-universal%20Teichmuller%20Theory.pdf`

[CbGT] Y. Hoshi, A. Minamide, S. Mochizuki, Group-theoreticity of Numerical Invariants and Distinguished Subgroups of Configuration Space Groups, *Kodai Math. J.* **45** (2022), pp. 295-348.

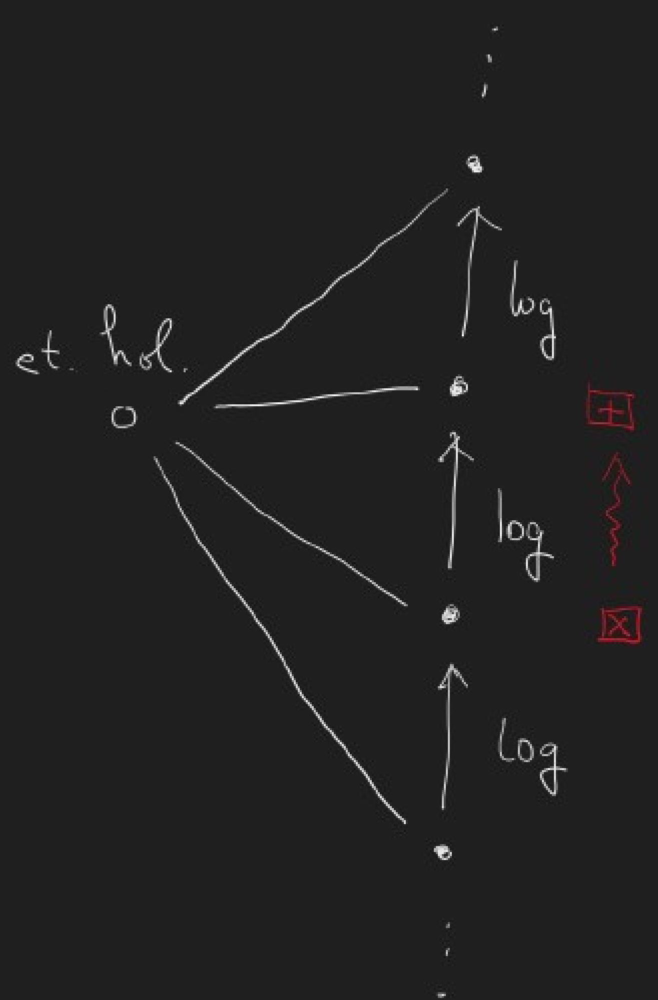
[CbGal] Y. Hoshi, S. Mochizuki, S. Tsujimura, Combinatorial construction of the absolute Galois group of the field of rational numbers, *J. Math. Sci. Univ. Tokyo.* **32** (2025), pp. 1-125.

[RNSPM] S. Mochizuki, S. Tsujimura, *Resolution of Nonsingularities, Point-theoreticity, and Metric-admissibility for p-adic Hyperbolic Curves*, RIMS Preprint **1974** (June 2023).

[ArGT] S. Mochizuki, S. Tsujimura, *Topics surrounding the arithmeticity of the Grothendieck-Teichmüller Group*, manuscript in preparation.

[Wtbe] H. Watanabe, Belyi injectivity for outer representations on certain quotients of étale fundamental groups of hyperbolic curves of genus zero, *Hiroshima Math. J.* **53** (2023), pp. 63-85.

Sequence of log-links in IUT



Tower of configuration space groups and GT